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CLASSICAL MODELS OF THE QUEUING THEORY AND GENERALIZATIONS

On the Nonstationary Erlang Loss Model[1]

A. I. Zeifman

Vologda State Pedagogical University,
Institute of Informatics Problems, Russian Academy of Sciences,
and Vologda Science-Coordinating Center CEMI, Russian Academy of Sciences, Vologda, Russia

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Abstract—Nonstationary loss queueing system (Erlang model) is considered. We study weak ergodicity, bounds on the rate of convergence, approximations, bounds for limit characteristics.

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1. INTRODUCTION

In this paper the queueing system $M/M/N/N$ is examined, which is called the Erlang model (Erlang first introduced and examined it in [1]) or the loss system (in Russian literature this system is sometimes denoted as $M/M/N/0$).

It is assumed for the general (nonstationary) case, that a request arrival intensity is $\lambda(t)$ and a service rate of each server is $\mu(t)$, the total number of servers is N, and the requests, that arrive when all servers in the system are busy, are lost. In this case the number of requests in the system (the queue length) $X(t)$ is a birth and death process (hereinafter referred to as BDP) with the state space $E_N = 0, 1, \ldots, N$ and intensities of birth $\lambda_k(t) = \lambda(t)$ and death $\mu_k(t) = k\mu(t)$. Having that, it is also assumed that the functions $\lambda(t)$ and $\mu(t)$ are locally integrable on $[0, \infty)$.

This paper is to a considerable degree a survey of the main results, obtained so far. Besides, some new bounds and constructions are introduced.

Recall the basic notations. Let $p_j(t) = \Pr\{X(t) = j\}$, $j = 0, 1, \ldots, N$ be the state probabilities for the BDP $X(t)$, and $\mathbf{p}(t) = (p_0(t), p_1(t), \ldots, p_N(t))^{\mathrm{T}}$. Then the process dynamics is described by the direct Kolmogorov equation system

$$\frac{d\mathbf{p}}{dt} = A\,(t)\,\mathbf{p}, \tag{1}$$

where the transfer rate matrix $A(t)$ is tridiagonal and the sum of elements of each column is equal to zero (for all $t \geqslant 0$). The overdiagonal and subdiagonal elements of the matrix are $\mu_k(t)$ and $\lambda_k(t)$ accordingly.

The BDP $X(t)$ is called *weakly ergodic*, if $\mathbf{p}^*(t) - \mathbf{p}^{**}(t) \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0)$ and $\mathbf{p}^{**}(0)$, and it is called (*strongly*) *ergodic*, if there exists a constant vector π (it is called a stationary distribution), such that for any $\mathbf{p}^*(0)$ it holds, that $\pi - \mathbf{p}^*(t) \to 0$ as $t \to \infty$.

In the early studies (see, for instance, references in [2]) the stationary (homogeneous) case was examined, when the arrival and service intensities were time-independent. In that case only strong ergodicity is possible, and moreover for this property to hold it is obviously necessary and sufficient, that the following condition $\lambda + \mu > 0$ to be met. The major subject of research here was (and still remains!) the so called convergence parameter $\beta = \beta(\lambda, \mu, N)$ (spectral lacuna), that is the modulus

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of the largest nonzero (negative) eigenvalue of the matrix A. It defines the rate of convergence to the stationary distribution. In a number of studies there were examined the properties of $\beta(\lambda, \mu, N)$ as a function of arrival and service intensities, as well as the number of servers in the system, see for example [3], and the current state of the art of this direction is described in [4]. In the past decade there have become more intense the studies on asymptotics of the convergence parameter of the stationary Erlang model for the case, when the requests arrival intensity is proportional to the number of servers, i.e., $\lambda = Nc$, $c > 0$, and the number itself $N \to \infty$, see [5–7]. The final results for this case were obtained quite recently in [8], see also a review of different research methods in [9]. Notably, they managed to prove that if $c < \mu$, then $\lim_{N\to\infty} \beta(Nc, \mu, N) = \mu$, if $c = \mu$, then $\beta(Nc, \mu, N) = 2\mu$, if finally $c > \mu$, then $\lim_{N\to\infty} \frac{\beta(Nc,\mu,N)}{N} = (\sqrt{c} - \sqrt{\mu})^2$.

Studies on the nonstationary Erlang model were begun in the 1970s, see [10]. For such models the properties of strong (see [11–13]) and weak [14, 15] ergodicity were examined, and later there were examined other characteristics, including steadiness, approximations, etc. The necessary and sufficient condition of ergodicity was obtained in [14] (see also [4]), it is that the integral $\int_0^\infty (\lambda(t) + \mu(t))\, dt$ diverges. The approach, that was used to study this model and more general nonstationary Markov queueing models, was first proposed in [10] by Gnedenko and Markov and later was developed in previous works of the author of this paper [16–19]. This approach is based on the notion and bounds, related to the logarithmic norm of an operator function, and on the special transforms of the reduced intensity matrix for the considered Markov chain, that allows to obtain explicit and precise bounds (see Section 2).

2. RESEARCH METHOD

Here we describe briefly the general approach. First, having that $\sum_{i=0}^{N} p_i(t) = 1$ for all $t \geqslant 0$, let $p_0(t) = 1 - \sum_{i=1}^{N} p_i(t)$, then we'll get the following system

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{2}$$

where

$$B = \begin{pmatrix} -(2\lambda+\mu) & (2\mu-\lambda) & -\lambda & \cdots & \cdots & -\lambda \\ \lambda & -(\lambda+2\mu) & 3\mu & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & \lambda & -(\lambda+N\mu) \end{pmatrix},$$

$$\mathbf{z} = (p_1, p_2, \ldots, p_N)^{\mathrm{T}}, \quad \mathbf{f} = (\lambda, 0, \ldots, 0)^{\mathrm{T}}.$$

Consider a homogeneous system, that corresponds to (2):

$$\frac{d\mathbf{u}(t)}{dt} = B(t)\mathbf{u}(t), \quad \mathbf{u} = (u_1, \ldots, u_N)^{\mathrm{T}}. \tag{3}$$

Then

$$\mathbf{p}^*(t) - \mathbf{p}^{**}(t) = \begin{pmatrix} \sum_{i=1}^{N} (u_i^{**}(t) - u_i^*(t)) \\ \mathbf{u}^*(t) - \mathbf{u}^{**}(t) \end{pmatrix}, \tag{4}$$

and, therefore, all the problems, related to ergodicity and convergence rate issues for the BDP $X(t)$, may be reduced to studying the system (4).

Let $d_1, \ldots, d_N$ be some positive numbers. Consider a triangular matrix

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots & d_1 \\ 0 & d_2 & d_2 & \cdots & d_2 \\ 0 & 0 & d_3 & \cdots & d_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d_N \end{pmatrix} \tag{5}$$

and set $\mathbf{v}(t) = D\mathbf{u}(t)$. Then

$$\frac{d\mathbf{v}(t)}{dt} = B^*(t)\mathbf{v}(t), \tag{6}$$

where the matrix

$$B^* = DBD^{-1} = \begin{pmatrix} -(\lambda+\mu) & \frac{d_1}{d_2}\mu & 0 & \cdots & 0 \\ \frac{d_2}{d_1}\lambda & -(\lambda+2\mu) & \frac{d_2}{d_3}2\mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(\lambda+(N-1)\mu) & \frac{d_{N-1}}{d_N}(N-1)\mu \\ 0 & 0 & \cdots & \frac{d_N}{d_{N-1}}\lambda & -(\lambda+N\mu) \end{pmatrix}. \tag{7}$$

Further it is the system (6) that will be examined. Its special form (the tridiagonal matrix of the system has nonnegative off-diagonal elements) makes it possible to use the bounds, obtained with the use of logarithmic norm. (This notion was introduced simultaneously and independently of each other by Lozinski [20] and Dahlquist [21] as an evaluator of the numerical integration error for an ordinary differential system, see also [22–24]. In [24] a similar notion for equations in Banach space is introduced, for more details see [8]). Notable, let us introduce the following quantities

$$\alpha_i(t) = \lambda(t) + i\mu(t) - \frac{d_{i+1}}{d_i}\lambda(t) - \frac{d_{i-1}}{d_i}(i-1)\mu(t), \quad i = 1, 2, \ldots, N, \tag{8}$$

where $d_0 = d_{N+1} = 0$, and assume:

$$\beta_*(t) = \min_i \alpha_i(t), \quad \beta^*(t) = \max_i \alpha_i(t). \tag{9}$$

We denote by $U(t,s)$ and $V(t,s)$ the Cauchy matrices for systems (3) and (6) correspondingly.

We'll make the following denotation $\|\mathbf{x}\| = \|\mathbf{x}\|_1 = \sum_i |x_i|$, set $\|x\|_{1D} = \|Dx\|$, and we'll use the same designations for the corresponding matrix forms. Then

$$\|V(t,s)\| = \|U(t,s)\|_{1D} \leqslant e^{-\int_s^t \beta_*(\tau)\,d\tau}, \tag{10}$$

and therefore

$$\|\mathbf{v}(t)\| = \|\mathbf{u}(t)\|_{1D} \leqslant e^{-\int_s^t \beta_*(\tau)\,d\tau} \|\mathbf{v}(s)\|, \tag{11}$$

and, moreover, if all components of vector $\mathbf{v}(s)$ are nonnegative, then

$$\|\mathbf{v}(t)\| \geqslant e^{-\int_s^t \beta^*(\tau)\,d\tau} \|\mathbf{v}(s)\|. \tag{12}$$

As a result the problem of obtaining the most accurate bounds is related to finding an auxiliary sequence $\{d_i\}$, such that functions in (9) were close to each other at most. As shown in [8], in the case of constant (time-independent) intensities this auxiliary sequence can be chosen in such way, that $\beta_* = \beta^*$, and thus these quantities are equal to the convergence parameter.

3. WEAK ERGODICITY AND BOUNDS

Denote by $E_k(t) = E\{X(t) \mid X(0) = k\}$ the average number of requests in system at a moment t under condition, that at the time zero there are k requests in the system. Sometimes we'll use also a more general expression $E_{\mathbf{p}}(t) = \sum_{k \geqslant 0} E\{X(t) \mid X(0) = k\} p_k(0)$.

Definition. Let us say, that a BDP $X(t)$ has a *limiting average* $\varphi(t)$, if

$$\lim_{t \to \infty} (\varphi(t) - E_k(t)) = 0$$

for any k.

Let us give the general statement first.

Theorem 1. *Let the sequence $\{d_i\}$ of positive numbers be such, that*

$$\int_0^\infty \beta_*(t)\, dt = +\infty. \tag{13}$$

Then the BDP $X(t)$ is weakly ergodic and has a limiting average and at any initial conditions the following bounds are valid:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leqslant \frac{4G}{d}\, e^{-\int_s^t \beta_*(\tau)\, d\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|, \quad 0 \leqslant s \leqslant t, \tag{14}$$

$$|E_{\mathbf{p}^*}(t) - E_{\mathbf{p}^{**}}(t)| \leqslant \frac{4G}{\omega}\, e^{-\int_s^t \beta_*(\tau)\, d\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|, \quad 0 \leqslant s \leqslant t, \tag{15}$$

$$\underline{\lim}_{t \to \infty} \Pr(X(t) < k) \geqslant 1 - \frac{d_1}{\sum_{j=1}^k d_i} \overline{\lim}_{t \to \infty} \int_0^t \lambda(u) e^{-\int_u^t \beta_*(\tau)\, d\tau} du, \tag{16}$$

$$\overline{\lim}_{t \to \infty} E_{\mathbf{p}}(t) \leqslant \frac{1}{W} \overline{\lim}_{t \to \infty} \int_0^t \lambda(u) e^{-\int_u^t \beta_*(\tau)\, d\tau} du, \tag{17}$$

where

$$G = \sum_{i=1}^N d_i, \quad d = \min d_i, \quad \omega = \min \frac{d_k}{k}, \quad W = \min \frac{\sum_{i=1}^k d_i}{k}.$$

The proof of Theorem 1 is given in the Appendix.

Remark 1. As was shown in [14], for the existence of the described sequence $\{d_i\}$ it is necessary and sufficient, that $\int_0^\infty (\lambda(t) + \mu(t))\, dt = +\infty$.

Remark 2. The bounds (16), (17) are new. They make the corresponding results in [25] more precise.

Remark 3. It is also possible to obtain from (16) an estimate of the limiting fault probability, viz. in terms of Theorem 1 we have

$$\overline{\lim}_{t \to \infty} \Pr(X(t) = N) \leqslant \frac{d_1}{\sum_{j=1}^N d_i} \overline{\lim}_{t \to \infty} \int_0^t \lambda(u) e^{-\int_u^t \beta_*(\tau)\, d\tau} du. \tag{18}$$

Note, that in the stationary case the study of the limiting fault probability for a most interesting and complex case when ($\lambda = Nc$, $c \leqslant \mu$) was held in [26], where bounds in the following form were obtained $\lim_{t\to\infty} \Pr(X(t) = N) \leqslant Cq^N$, $q < 1$, when $c < \mu$, and $\lim_{t\to\infty} \Pr(X(t) = N) < DN^{-1/2}$ when $c = \mu$.

Remark 4. When studying particular queueing systems, described by the considered model, there appears an interesting problem of choice of the best auxiliary sequence $\{d_i\}$ in the sense, that bounds (16), (17), (18) should be precise at most.

Consider now several particular cases. First, let the arrival and the service intensities be periodical. For the sake of convenience we'll assume that the length of a period is equal to one.

Corollary 1. *If arrival and service intensities are 1-periodic, then for the weak ergodicity of the process it is necessary and sufficient, that* $\int_0^1 (\lambda(t) + \mu(t))\, dt > 0$. *Under the preceding condition there exists a limiting 1-periodic regime of distribution of the state probabilities of the process* $\pi(t) = (\pi_0(t), \pi_1(t), \ldots, \pi_N(t))^{\mathrm{T}}$, *a corresponding 1-periodic limiting average* $\phi(t) = \sum_{k=0}^{N} k\pi_k(t)$, *and the following bounds on the rate of convergence are valid, instead of* (14) and (15):

$$\|\mathbf{p}^*(t) - \pi(t)\| \leqslant \frac{8G}{d} e^{-\int_0^t \beta_*(\tau)\, d\tau}, \tag{19}$$

$$|E_{\mathbf{p}^*}(t) - \phi(t)| \leqslant \frac{8G}{\omega} e^{-\int_0^t \beta_*(\tau)\, d\tau}. \tag{20}$$

Let us return to the general case now.

Corollary 2. *Assume the following condition holds*

$$\int_0^\infty \mu(t)\, dt = +\infty. \tag{21}$$

Then, assuming that all $d_i = 1$, *we obtain the following bounds*:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leqslant 4N e^{-\int_s^t \mu(\tau)\, d\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|, \quad 0 \leqslant s \leqslant t, \tag{22}$$

$$|E_{\mathbf{p}^*}(t) - E_{\mathbf{p}^{**}}(t)| \leqslant 4N^2 e^{-\int_s^t \mu(\tau)\, d\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|, \quad 0 \leqslant s \leqslant t, \tag{23}$$

$$\varliminf_{t\to\infty} \Pr(X(t) < k) \geqslant 1 - \frac{1}{k} \varlimsup_{t\to\infty} \int_0^t \lambda(u) e^{-\int_u^t \mu(\tau)\, d\tau} du, \tag{24}$$

$$\varlimsup_{t\to\infty} E_{\mathbf{p}}(t) \leqslant \varlimsup_{t\to\infty} \int_0^t \lambda(u) e^{-\int_u^t \mu(\tau)\, d\tau} du. \tag{25}$$

The next statement can be of interest in the case of large intensity of requests' arrival.

Corollary 3. *Let the following condition hold for some* $\delta \in (0, 1)$

$$\int_0^\infty (\delta\lambda(t) - N\mu(t))\, dt = +\infty. \tag{26}$$

Then as $d_k = \delta^{k-1}$, $k = 1, \dots, N$, *the bounds* (14)–(17) *are valid, where* $G = \frac{1-\delta^N}{1-\delta}$, $d = \delta^{N-1}$, $\omega = \frac{\delta^{N-1}}{N}$, $W = \min \frac{\sum_{i=0}^{k-1} \delta^i}{k}$, *and* $\beta_*(t) = (\delta^{-1} - 1)(\delta\lambda(t) - N\mu(t))$.

Remark 5. It's easy to see, that for the stationary Erlang model the condition from Corollary 3 is equivalent to $\lambda > N\mu$. Then, having chosen $\delta = \sqrt{\frac{N\mu}{\lambda}}$, we obtain an asymptotically exact (as $N \to \infty$ and with λ/N fixed, see [8]) value of the convergence parameter $\beta = \left(\sqrt{\lambda} - \sqrt{N\mu}\right)^2$.

Remark 6. Bounds related to the model robustness are not considered in this paper. The corresponding results can be found in [27], see also [4].

4. APPROXIMATION

In the case when system (1) is of high dimension (i.e., the number of servers N is large) to numerically find the limiting characteristics of the process there appears a problem of approximation of the system with similar systems of lesser dimension. Problems of this sort were examined for countable BDPs in [28] and later in [19]. First crude bounds for a large finite state space, based on simple transforms of system (1), are given in [4]. The approach that allows to obtain more accurate bounds is based on using the method, described in Section 2.

Consider a family of "reduced" BDPs $X_n(t)$ with sets of states $E_n = \{0, 1, \dots, n\}$. We'll denote the corresponding characteristics of reduced processes by index n, and vectors of the form $(x_0, x_1, \dots, x_n)^{\mathrm{T}}$ of length $n+1$ and $(x_0, x_1, \dots, x_n, 0, \dots, 0)^{\mathrm{T}}$ of length $N+1$ will be considered identical in current section. Let us rewrite the system (1) in the following form

$$\frac{d\mathbf{p}}{dt} = A_n(t)\,\mathbf{p} + (A(t) - A_n(t))\,\mathbf{p}. \tag{27}$$

If we now choose $\mathbf{p}(0) = \mathbf{p}_n(0)$, then we'll get:

$$\begin{aligned}\mathbf{p}(t) &= W_n(t,0)\mathbf{p}(0) + \int_0^t W_n(t,\tau)\,(A(\tau) - A_n(\tau))\,\mathbf{p}(\tau)\,d\tau \\ &= \mathbf{p}_n(t) + \int_0^t W_n(t,\tau)\,(A(\tau) - A_n(\tau))\,\mathbf{p}(\tau)\,d\tau,\end{aligned} \tag{28}$$

where $W(t,s)$ is the Cauchy matrix of system (1). For the reason, that $\|W(t,s)\| \leqslant 1$ we have:

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| \leqslant \int_0^t \|\,(A(\tau) - A_n(\tau))\,\mathbf{p}(\tau)\|\,d\tau. \tag{29}$$

Then, taking into consideration, that the rows of the matrix $A(t) - A_n(t)$ with numbers $0, 1, \dots, n-1$ are null equations, we get

$$\|\,(A(\tau) - A_n(\tau))\,\mathbf{p}(\tau)\| \leqslant \|A(\tau) - A_n(\tau)\| \sum_{k=n}^{N} p_k(\tau), \tag{30}$$

and then

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| \leqslant 2\,(\Lambda(t) + N\Psi(t)) \int_0^t \sum_{k=n}^{N} p_k(\tau)\,d\tau, \tag{31}$$

where

$$\Lambda(t) = \operatorname{ess} \sup_{0 \leqslant s \leqslant t} \lambda(s), \quad \Psi(t) = \operatorname{ess} \sup_{0 \leqslant s \leqslant t} \mu(t), \tag{32}$$

so to assess the reduction error it's sufficient to make an estimate of the value $\sum_{k=n}^{N} p_k(\tau)$.

Theorem 2. *Let $\{d_i\}$ be a an arbitrary sequence of positive numbers, and at the initial time the system is empty (i.e., $X(0) = X_n(0) = 0$). Then for all $t \geqslant 0$, $n \leqslant N$ the following estimate is valid*:

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| \leqslant \frac{2d_1\left(\Lambda(t) + N\Psi(t)\right)}{\sum\limits_{j=1}^{n} d_j} \int\limits_0^t du \int\limits_0^u \lambda(\tau) e^{-\int\limits_\tau^u \beta_*(s)\, ds} d\tau. \tag{33}$$

The proof of Theorem 2 is given in the Appendix.

Remark 7. Let, for example, $\lambda(t) \leqslant \mu(t)$ for all $t \geqslant 0$. Then, assuming $d_k = 2^{k-1}$, $k \geqslant 1$, we obtain:

$$\alpha_i(t) = \lambda(t) + i\mu(t) - \frac{d_{i+1}}{d_i}\lambda(t) - \frac{d_{i-1}}{d_i}(i-1)\mu(t) \leqslant \mu(t) - \lambda(t), \tag{34}$$

whence it follows, that $\beta_*(t) \geqslant 0$ and Theorem 2 allows to get a simple estimate:

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| \leqslant \frac{2\,(N+1)\,\Psi(t)}{2^n - 1} \int\limits_0^t du \int\limits_0^u \lambda(\tau)\, d\tau \leqslant \frac{(N+1)\, t^2 \Psi^2(t)}{2^n - 1}, \tag{35}$$

where the right side rather quickly tends to zero for any fixed t and as n increases.

APPENDIX

Proof of Theorem 1. First, note that it follows from (4) that

$$\|\mathbf{u}^* - \mathbf{u}^{**}\| \leqslant \|\mathbf{p}^* - \mathbf{p}^{**}\| \leqslant 2\|\mathbf{u}^* - \mathbf{u}^{**}\|. \tag{A.1}$$

Further we have:

$$\|\mathbf{v}\| \leqslant \|D\|\|\mathbf{u}\| = \sum_{i=1}^{N} d_i \|\mathbf{u}\| = G\|\mathbf{u}\|, \quad \|\mathbf{u}\| \leqslant \|D^{-1}\|\|\mathbf{v}\| \leqslant \frac{2}{d}\|\mathbf{v}\|. \tag{A.2}$$

And therefore

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leqslant 2\|\mathbf{u}^*(t) - \mathbf{u}^{**}(t)\| \leqslant \frac{4}{d}\|\mathbf{u}^*(t) - \mathbf{u}^{**}(t)\|_{1D}$$
$$\leqslant \frac{4}{d} e^{-\int\limits_s^t \beta_*(\tau)\, d\tau} \|\mathbf{u}^*(s) - \mathbf{u}^{**}(s)\|_{1D} \leqslant \frac{4G}{d} e^{-\int\limits_s^t \beta_*(\tau)\, d\tau} \|\mathbf{u}^*(s) - \mathbf{u}^{**}(s)\|, \tag{A.3}$$

whence the first estimate follows.

Introduce $\|\mathbf{x}\|_E = \sum_{k=1}^{N} k|x_k|$, then

$$\|\mathbf{x}\|_E = \sum_{k=1}^{N} d_k |x_k| \frac{k}{d_k} \leqslant \omega^{-1} \sum_{k=1}^{N} d_k |x_k| \leqslant \frac{2}{\omega}\|\mathbf{x}\|_{1D}, \tag{A.4}$$

hence,

$$|E_{\mathbf{p}^*}(t) - E_{\mathbf{p}^{**}}(t)| \leqslant \frac{2}{\omega}\|\mathbf{u}^*(t) - \mathbf{u}^{**}(t)\|_{1D} \leqslant \frac{4G}{\omega} e^{-\int\limits_s^t \beta_*(\tau)\,d\tau} \|\mathbf{u}^*(s) - \mathbf{u}^{**}(s)\|, \tag{A.5}$$

whence the second estimate follows.

From system (2) we obtain in $1D$-norm:

$$\|\mathbf{z}(t)\| = \left\| V(t,0)\mathbf{z}(0) + \int\limits_0^t V(t,\tau)\,\mathbf{f}(\tau)\,d\tau \right\| \leqslant e^{-\int\limits_0^t \beta_*(\tau)\,d\tau} \|\mathbf{z}(0)\| + d_1 \int\limits_0^t \lambda(u) e^{-\int\limits_u^t \beta_*(\tau)\,d\tau} du, \tag{A.6}$$

and, hence,

$$\overline{\lim_{t\to\infty}} \|\mathbf{z}(t)\|_{1D} \leqslant d_1 \overline{\lim_{t\to\infty}} \int\limits_0^t \lambda(u) e^{-\int\limits_u^t \beta_*(\tau)\,d\tau} du. \tag{A.7}$$

The author of this paper is only interested in nonnegative solutions of system (2). For such solutions we have:

$$\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\| = d_1 p_1 + (d_1 + d_2)p_2 + \ldots + (d_1 + \ldots + d_N)p_N \tag{A.8}$$

and, hence,

$$\sum_{i=k}^{N} p_k \leqslant \frac{1}{\sum\limits_{j=1}^{k} d_j} \left(\sum_{j=1}^{k} d_j p_k + \ldots + \sum_{j=1}^{N} d_j p_N \right) \leqslant \frac{1}{\sum\limits_{j=1}^{k} d_j} \|\mathbf{z}\|_{1D}, \tag{A.9}$$

whence

$$\overline{\lim_{t\to\infty}} \sum_{i=k}^{N} p_k(t) \leqslant \frac{d_1}{\sum\limits_{j=1}^{k} d_j} \overline{\lim_{t\to\infty}} \int\limits_0^t \lambda(u) e^{-\int\limits_u^t \beta_*(\tau)\,d\tau} du, \tag{A.10}$$

where the third estimate follows from.

At last, the fourth estimate follows from the inequality

$$\sum_{k=1}^{N} k p_k \leqslant \frac{1}{W} \sum_{k=1}^{N} \sum_{j=1}^{k} d_j p_k = \frac{1}{W} \|\mathbf{z}\|_{1D}. \tag{A.11}$$

Theorem 1 is proved.

Proof of Theorem 2. We obtain the following estimate from inequalities (A.6) and (A.9):

$$\sum_{i=k}^{N} p_k \leqslant \frac{d_1}{\sum\limits_{j=1}^{k} d_j} \int\limits_0^t \lambda(u) e^{-\int\limits_u^t \beta_*(\tau)\,d\tau} du. \tag{A.12}$$

To finish the proof one has to use inequality (31).

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